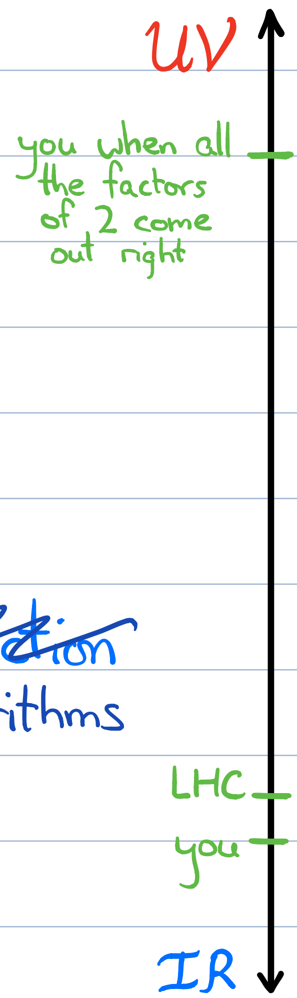
Renormalizing the Renormalization Group



OR

How I Learned to Stop Worrying and Love ~~the β Function~~ Logarithms

References:

- David Skinner's "Advanced QFT" (ch. 5)
- Francois David's "QFT II"
- Seth Koren's PhD thesis :)
- Polchinski, "Renormalization and Effective Lagrangians"
- A 50-page-long QFT II problem set of Frederik Denef
- QFT texts: P&S, Srednicki, Zee, Schwartz

Conventions:
- $\hbar = c = k_B = G_N \equiv 1$.
- $\mathbb{N}$ includes zero.
- $\log$ is the natural logarithm.
- $M = (\mathbb{R}^d, \delta)$

flat space, Euclidean signature

[$|\vec{p}| = 0 \iff \vec{p} = \vec{0}$ avoids lightcone issues with $|\vec{p}| < \Lambda$]

Motivation: A man falling, whether from a bridge or in love, quickly forgets the minutiae of the life he left behind. He feels only the all-consuming madness of existence and a quickening heartbeat.

$\boxed{0.}$ Wilsonian RG: setup & philosophy

An experimentalist gets you drunk. You stumble into a Euclidean theory with action $S_{\Lambda_0}[\phi]$,

read: Hamiltonian

where $\phi(x) = \int_{\mathbb{R}^d} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x} \tilde{\phi}(p)$ has momenta below Λ_0:

$$\tilde{\phi}(p) = 0 \text{ for } |p| > \Lambda_0 \iff \operatorname{supp}(\tilde{\phi}) \subset B^d(\Lambda_0).$$

Define the quantum theory by $Z_{\Lambda_0} = \int_0^{\Lambda_0} D\phi \, e^{-S_{\Lambda_0}[\phi]}$,

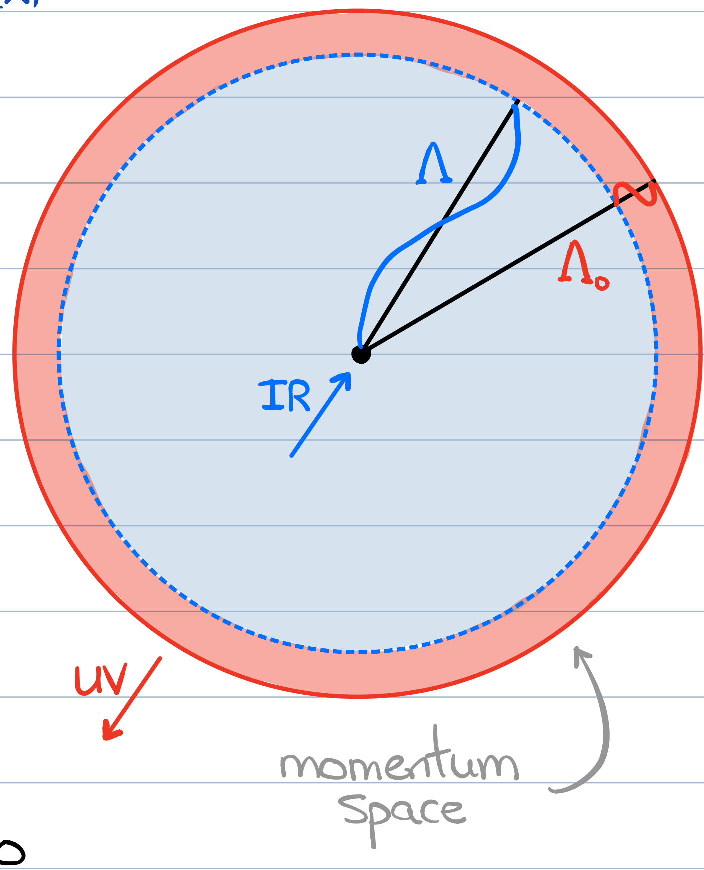
where $\int_0^{\Lambda_0}$ recalls the range of momenta of $\tilde{\phi}$.

Split up $\phi(x) = \phi_s(x) + \phi_f(x)$ into slow & fast modes:

$$\phi_s(x) = \int_0^{\Lambda} \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot x}\, \tilde{\phi}_s(p)$$

$$\phi_f(x) = \int_{\Lambda}^{\Lambda_0} \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot x}\, \tilde{\phi}_f(p)$$

where $\Lambda = s\Lambda_0 < \Lambda_0$ ($s = 1 - \delta s$, $0 < \delta s \ll 1$).



Then $D\phi = D\phi_s\, D\phi_f$, so

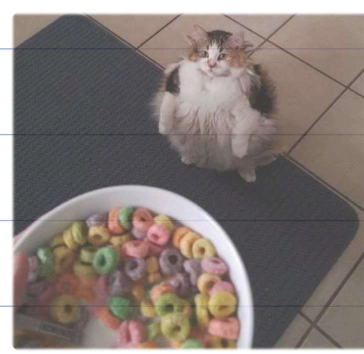
$$Z_{\Lambda_0} = \int_0^{\Lambda_0} D\phi\, e^{-S_{\Lambda_0}[\phi]} = \int_0^{\Lambda} D\phi_s \underbrace{\int_{\Lambda}^{\Lambda_0} D\phi_f\, e^{-S_{\Lambda_0}[\phi_s+\phi_f]}} = $$

$$= \int_0^{\Lambda} D\phi_s\, e^{-S^{\text{eff}}_{\Lambda}[\phi_s]} = Z_{\Lambda}.$$

Here $e^{-S^{\text{eff}}_{\Lambda}[\phi_s]} \equiv \int_{\Lambda}^{\Lambda_0} D\phi_f\, e^{-S_{\Lambda_0}[\phi_s+\phi_f]}$; equivalently,

$$\boxed{S^{\text{eff}}_{\Lambda}[\phi_s] = -\log\left[\int_{\Lambda}^{\Lambda_0} D\phi_f\, e^{-S_{\Lambda_0}[\phi_s+\phi_f]}\right].}$$

In practice, one can do the path integral over ϕ_f only perturbatively, using lööps, counterterms, etc. F.



Observations: ① $Z_{\Lambda_0} = Z_\Lambda$, since doing a path integral doesn't change its value.

② $S^{\text{eff}}_\Lambda \neq S_{\Lambda_0}$: the couplings g_i depend on Λ, i.e. they "run": $g_i = g_i(\Lambda)$.

[This is because interactions among the modes previously mediated by fluctuations in the ϕ_f must now be supplied to S^{eff}_Λ directly.]

③ Given an initial scale Λ_0, the map $\Lambda \longmapsto S^{\text{eff}}_\Lambda[\phi]$ (given by integrating out momenta below $\Lambda < \Lambda_0$) defines the RENORMALIZATION SEMIGROUP FLOW on the moduli space $\mathcal{M}$ of all theories.

The plan:

1. Main tools: the RGE and β, γ
2. RG flow I: formal development
3. RG flow II: geometry & philosophy
4. Polchinski's version of Wilsonian RG

1. The Main Tools: the RGE and β, γ

Recall that $Z_{\Lambda_0} = \int_0^{\Lambda_1} D\phi_s \, e^{-S^{\text{eff}}_{\Lambda_0}[\phi_s]} = Z_{\Lambda_1}$.

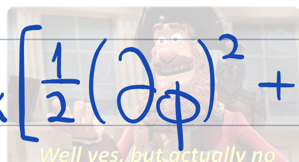
We can keep integrating out heavy d.o.f:

$Z_{\Lambda_0} = Z_{\Lambda_1} = Z_{\Lambda_2} = \dots$ [UV $\longrightarrow$ IR, or top-down]

"just do a bit of the PI"

$\longrightarrow$ Big idea: take Λ down infinitesimally from Λ_0.

[here $\phi \equiv \phi_s$] Z_Λ should remain constant. If



$$S^{\text{eff}}_\Lambda[\phi] = \int_{\mathbb{R}^d} d^d x \left[\tfrac{1}{2}(\partial\phi)^2 + \sum_{i=1}^n g_i(\Lambda)\, \mathcal{O}_i(\phi, \partial\phi)\right], \text{ then}$$

$$0 = \frac{dZ_\Lambda}{d(\log\Lambda)} = \Lambda \frac{d}{d\Lambda} Z_\Lambda(\Lambda, \{g_i(\Lambda)\}) =$$

$$= \boxed{\left(\Lambda \frac{\partial}{\partial\Lambda}\bigg|_{g_i} + \sum_{i=1}^n \Lambda \frac{\partial g_i}{\partial\Lambda} \frac{\partial}{\partial g_i}\bigg|_\Lambda\right) Z_\Lambda(\{g_i\}) = 0.} \quad \text{[Wilson ERGE]}$$

The couplings "run" to account for the changing d.o.f., and to ensure that Z_Λ is independent of scale. We introduce the β f$^{\underline{n}}$ to measure this:

$$\boxed{\beta_i(g_i) \equiv \frac{\partial g_i}{\partial(\log\Lambda)} = \Lambda \frac{\partial g_i}{\partial\Lambda}.}$$

But the kinetic "coupling" 1 can also run:

$$S^{\text{eff}}_\Lambda[\phi] = \int_{\mathbb{R}^d} d^d x \left[\frac{Z_\Lambda}{2}(\partial\phi)^2 + \sum_{i=1}^{n} g_i(\Lambda)\,\mathcal{O}_i(\phi, \partial\phi)\right].$$

This is annoying, so we renormalize ϕ at every scale Λ to keep it canonically normalized:

$$\varphi(x) \equiv \sqrt{Z_\Lambda}\,\phi(x).$$ ["wave function renormalization"]

$$\Downarrow$$

$$S^{\text{eff}}_\Lambda[\varphi] = \int_{\mathbb{R}^d} d^d x \left[\frac{1}{2}(\partial\varphi)^2 + \sum_{i=1}^{n} Z_\Lambda^{n_i/2} g_i(\Lambda)\,\mathcal{O}_i(\varphi, \partial\varphi)\right].$$

This normalizes φ at any scale Λ, but integrating out more modes will cause Z_Λ to reappear. The anomalous dimension γ_ϕ of ϕ is essentially the β function of its kinetic coupling:

$$\gamma_\phi = -\frac{1}{2}\Lambda\frac{\partial(\log Z_\Lambda)}{\partial\Lambda} = \beta\left(\log\left(Z_\Lambda^{1/2}\right)\right).$$

⚠ γ_ϕ and the β_i all depend implicitly on <u>all</u> of the couplings. The Callan-Symanzik equation ties all of these dependences together.

2. The RG Flow: formal development

2.1. The Callan-Symanzik Equation

Consider the n-point correlator $\Gamma^{(n)}_\Lambda(x_1,\dots,x_n;\ \Lambda,\ g_i)$:

$$\Gamma^{(n)}_\Lambda \equiv \langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{\mathcal{Z}_\Lambda}\int_0^\Lambda D\phi\,\phi(x_1)\cdots\phi(x_n)\,e^{-S_\Lambda[Z_\Lambda^{1/2}\phi]}$$

$\phi \longrightarrow \varphi$

$$= \frac{1}{\mathcal{Z}_\Lambda}\int_0^\Lambda D\varphi\, Z_\Lambda^{-n/2}\,\varphi(x_1)\cdots\varphi(x_n)\,e^{-S_\Lambda[\varphi]} =$$

$$= Z_\Lambda^{-n/2}\langle\varphi(x_1)\cdots\varphi(x_n)\rangle \equiv Z_\Lambda^{-n/2}\,\overline{\Gamma}^{(n)}_\Lambda.$$

Like $\mathcal{Z}_\Lambda$, the $\Gamma^{(n)}_\Lambda$ should be unchanged when we do part of the path integral: if the $\phi(x_j)$ insertions have momenta far below Λ, then $\Gamma^{(n)}_{\Lambda_0} = \Gamma^{(n)}_\Lambda$:

$$Z_{\Lambda_0}^{-n/2}\,\overline{\Gamma}^{(n)}_{\Lambda_0}(x_j;\Lambda_0, g_i(\Lambda_0)) = Z_\Lambda^{-n/2}\,\overline{\Gamma}^{(n)}_\Lambda(x_j;\Lambda, g_i(\Lambda)) \Longrightarrow$$

$$0 = \frac{d\Gamma^{(n)}_\Lambda}{d(\log\Lambda)} = \Lambda\frac{d}{d\Lambda}\Gamma^{(n)}_\Lambda(x_j;\ \Lambda,\ g_i(\Lambda)) =$$

$$= Z_\Lambda^{-n/2}\left(\Lambda\frac{\partial}{\partial\Lambda}\Big|_{g_i} + \sum_{i=1}^{n}\Lambda\frac{\partial g_i}{\partial\Lambda}\frac{\partial}{\partial g_i}\Big|_\Lambda - \frac{n}{2}\frac{1}{Z_\Lambda}\frac{\partial Z_\Lambda}{\partial\Lambda}\right)\overline{\Gamma}^{(n)}_\Lambda \Longrightarrow$$

$$\boxed{\Lambda\frac{d\Gamma^{(n)}_\Lambda}{d\Lambda} = \left(\Lambda\frac{\partial}{\partial\Lambda} + \sum_{i=1}^{n}\beta_i\frac{\partial}{\partial g_i} - n\gamma_\phi\right)\overline{\Gamma}^{(n)}_\Lambda = 0.}$$ [Callan-Symanzik]

2.2. Anomalous Dimensions

By now γ_φ has shown up twice: what is it?

To find out: ① integrate out fast modes, taking the cutoff down: $\Lambda \longrightarrow \Lambda' = s\Lambda$;

$\left[\begin{array}{l} \Lambda \longleftrightarrow \Lambda_0, \\ \Lambda' \longleftrightarrow \Lambda \end{array}\right]$

② rescale $x \longrightarrow x' \equiv sx$ to restore the cutoff: $\Lambda' = s\Lambda \longrightarrow \frac{s\Lambda}{s} = \Lambda$.

By Appendix A, $\varphi(x) \longrightarrow \bar{\varphi}(x) \equiv s^{\frac{d-2}{2}} \varphi(sx)$, so

$$\overline{\Gamma}^{(n)}_\Lambda(x_j; \Lambda, g_i(\Lambda)) = \left(\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \overline{\Gamma}^{(n)}_{s\Lambda}(x_j; \overset{①}{s\Lambda}, g_i(s\Lambda)) =$$

$\left[\text{Without } Z_\Lambda\text{, we'd expect } \Gamma^{(n)}_\Lambda(x) = s^{n(2-d)/2}\, \Gamma^{(n)}_\Lambda(sx)\right]$

$$= \left(\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} s^{n\left(\frac{2-d}{2}\right)} \overline{\Gamma}^{(n)}_\Lambda(sx_j; \overset{②}{\Lambda}, \overset{(!)}{g_i(s\Lambda)}).$$

Evidently integrating out $p \in [s\Lambda, \Lambda]$ modifies the scaling of φ: $\overline{\Gamma}^{(n)}_\Lambda$ behaves as if φ scaled as

$$\varphi(x) \longrightarrow \bar{\varphi}(x) \equiv \left[s^{2-d} \frac{Z_\Lambda}{Z_{s\Lambda}}\right]^{1/2} \varphi(sx) = s^{\Delta_\varphi} \varphi(sx).$$

If $s = 1 - \delta s$, then $s^{\Delta_\varphi} \simeq 1 + \Delta_\varphi \delta s$, while

$$\left[s^{2-d} \frac{Z_\Lambda}{Z_{s\Lambda}}\right]^{1/2} \simeq \left(1 - \frac{2-d}{2}\delta s\right)\left(1 + \frac{1}{2Z_\Lambda}\frac{\partial Z_\Lambda}{\partial \Lambda}\delta s\right) \simeq 1 + \left[\frac{d-2}{2} + \gamma_\varphi\right]\delta s.$$

Thus the scaling dimension $\Delta_\varphi = \frac{d-2}{2} + \gamma_\phi$ gets quantum corrections, measured by γ_ϕ.

Comments: ⓪ The same analysis can be carried out for any local operator $\mathcal{O}$ in the action.

① We can compare $S^{\text{eff}}_{\Lambda'}$ directly to S_Λ by rescaling $\Lambda' = s\Lambda \longrightarrow \frac{s\Lambda}{s} = \Lambda$. This introduces $s^{\Delta_{\mathcal{O}_i}}$ at each $\mathcal{O}_i$. We view these scalings as RG flows of the g_i, which may get stronger or weaker with Λ. This Λ-dependence is all $\Delta_{\mathcal{O}_i}$'s fault. To wit:

② $\Delta_{\mathcal{O}_i} > 0 \implies$ IR-suppressed: irrelevant $[g_i^I]$
$\Delta_{\mathcal{O}_i} < 0 \implies$ IR-enhanced : relevant $[g_i^R]$
$\Delta_{\mathcal{O}_i} = 0 \implies$ unchanged w/ Λ: marginal $[g_i^M]$

actually another lie lol

③ The presence of γ_ϕ makes $\Delta_{\mathcal{O}_i}$ depend nontrivially on Λ! The RG flow depends crucially on γ_ϕ.

④ Marginal operators don't run: $\beta(g) \equiv 0$. An action composed of such $\mathcal{O}$ is scale-invariant, and is a fixed point of the RG flow.

2.3. Fixed Points of the RG Flow

E.g. Consider $\Gamma_\Lambda^{(2)}$ at a fixed point:

$$0 = \left(\Lambda \frac{\partial}{\partial \Lambda} + \sum_{i=1}^{n} \cancelto{0}{\beta_i(g_i(\Lambda))} \frac{\partial}{\partial g_i} - 2\gamma_\phi(\cancel{\Lambda})\right)\Gamma_\Lambda^{(2)} \Longrightarrow$$

$$\Lambda \frac{\partial \Gamma_\Lambda^{(2)}}{\partial \Lambda} = 2\gamma_\phi^* \Gamma_\Lambda^{(2)} \Longrightarrow \Gamma_\Lambda^{(2)}(x,y) \propto \frac{c(g_i^*)}{|x-y|^{2\Delta_\phi}}.$$

So $\beta \equiv 0$ fixes the 2-pt f$^{\underline{n}}$ of a scale-invariant theory (CFT), and γ_ϕ gives $\Gamma_\Lambda^{(2)}$ the right scaling. One can also show that this implies $\langle T^\mu_\mu \rangle \overset{!}{=} 0$.

E.g. Near a fixed point, $g_i = g_i^* + \delta g_i$, so

$$\beta_i(g_i) = \beta_i(g_i^* + \delta g_i) \simeq \cancelto{0}{\beta_i(g_i^*)} + \left.\frac{\partial \beta_i}{\partial g_j}\right|_{g_i^*} \delta g_j = B_{ij}\,\delta g_j .$$

- The matrix $B_{ij} = \left.\frac{\partial \beta_i}{\partial g_j}\right|_{g_i^*}$ acts on the moduli space $\mathcal{M}$ of all actions, parametrized by their couplings g_i.

- B_{ij} is a linearized RG flow; its off-diagonal entries cause operator mixing. $\exists$ "dangerously irrelevant" operators that get negative corrections due to γ_ϕ and become relevant in the IR.

- The "eigen-operators" σ_i of B_{ij} do not mix, and have eigenvalues Δ_{σ_i}: $\sigma_i(\Lambda) = (\Lambda/\Lambda_0)^{\Delta_{\sigma_i}} \sigma_i(\Lambda_0)$.

- Can $B_{ij} = \frac{\partial \beta_i}{\partial g_j}\Big|_{g_i^*}$ be diagonalized? Not always!

When $B_{ij} = B_{ji}$, the VF $\beta_i(g_i)$ is curl-free, so* the β_i are a gradient! The potential is called $c(g_i)$, and $c(g_i^*)$ is the central charge of the resulting CFT.

Thm (Zamolodchikov). There is a c-function in any unitary, Lorentz-invariant QFT in 2 dimensions.

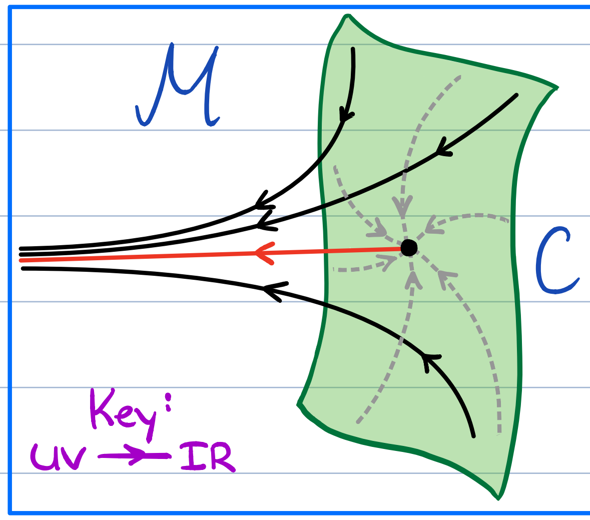
N.B. RG flow $\Longleftrightarrow$ gradient flow (Zam.)
$\Longleftrightarrow$ heat flow (Polch.)

* depending on the topology of $\mathcal{M}$

3. The RG Flow: geometry & philosophy

3.1. The Geometry of Theory Space

$\exists$ infinitely many g_i^I $(\Delta_{O_i} > 0)$ [we can just add fields & derivatives], but only finitely many g_i^R & g_i^M $(\Delta_{O_i} \leq 0)$.
This allows us to draw $\mathcal{M}$:



The critical surface $C \subset \mathcal{M}$ is the span of the g_i^I, and has finite codimension in $\mathcal{M}$.

- C is like Vegas: RG flow suppresses the g_i^I, so any $S_{\Lambda_0} \in C$ will flow to an IR fixed point still on C.

- If $S_{\Lambda_0} \notin C$, relevant operators drive it away from C. The RG trajectory can do several things:

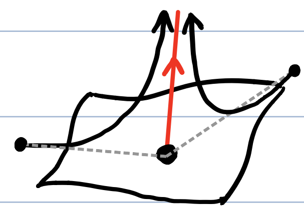
  1. Flow away from C forever
  2. Hit an IR fixed point at scale Λ^*:
     - trivial/Gaussian if all $g_i^*(\Lambda^*) = 0$,
     - nontrivial if not. [E.g. $\mathcal{N}=4$ SYM?]

③ Flow to a limit cycle!
④ Exhibit chaos, attractors, fractal behavior, etc.

3.2. Three Cool RG Behaviors

① Generically, $S_{\Lambda_0} \notin C$ and has both relevant & irrelevant couplings. The g_i^I quickly die off, and many S_Λ are repelled from C by the g_i^R.

Without the g_i^I, many S_Λ focus in the same relevant directions and become a single low-energy theory. This is universality.



[Amazingly, RG flow funnels all the details of a high-energy theory into a few effective parameters.]

② In QED, $e(\Lambda) = e_0 [1 - \beta_2 e_0 \log(\Lambda/m_e)]^{-1}$

diverges at $\Lambda_L = m_e \exp\left(\frac{1}{\beta_2 e_0}\right) \simeq 10^{286}$ GeV.

The theory does not exist for $\Lambda > \Lambda_L$, unless the observed electron charge is $e_0 = 0$. Such "quantum-trivial" theories always have Landau poles.

[Landau himself famously gave up on QED after proving that the electron doesn't exist.]

E.g. ③ Limit cycles in an RG flow yield a theory that looks the same at scales $\Lambda, \frac{\Lambda}{L}, \frac{\Lambda}{L^2}, \ldots$.

The Matryoshka doll Hamiltonian has β function

$$\beta(g) = \frac{dg}{d(\log \Lambda)} = g^2(\Lambda) + h^2.$$ Set $u \equiv \frac{g}{h}$, $t \equiv h \log \Lambda$.

Then $\frac{du}{dt} = u^2 + 1 \Longrightarrow u(t) = \tan[t + \tan^{-1} u_0] \Longrightarrow$

$$g(\Lambda) \sim h \tan(\Lambda + c).$$

Periodicity in Λ makes the system resemble nesting dolls.

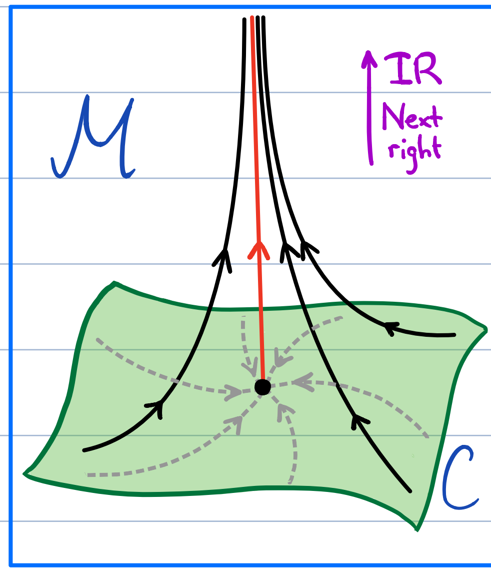
3.3. The Continuum Limit

→ Idea: watch the scale-Λ theory as we send the initial scale $\Lambda_0 \longrightarrow \infty$.

- S_{Λ_0} a CFT: $\Lambda_0 \longrightarrow \infty$ leaves $S_{\Lambda_0} = S_\Lambda$ unchanged.

- $S_{\Lambda_0} \in C$: raising Λ_0 leaves more room to flow down to scale Λ; as $\Lambda_0 \longrightarrow \infty$, S_Λ becomes a CFT.

- $S_{\Lambda_0} \notin C$: raising Λ_0 strengthens the g_i^R at scale Λ; as $\Lambda_0 \longrightarrow \infty$, the couplings diverge...

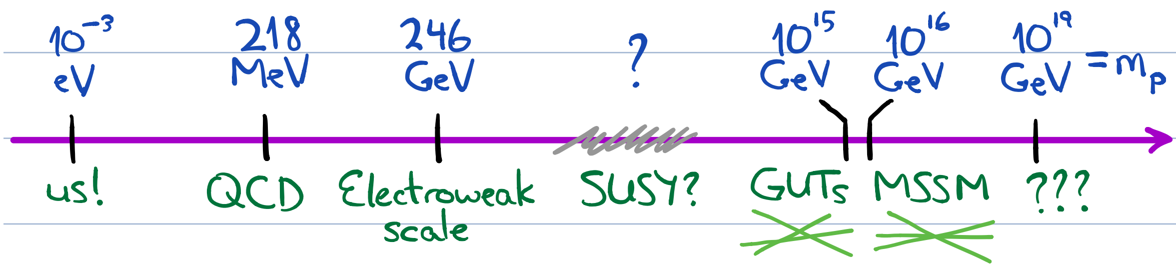


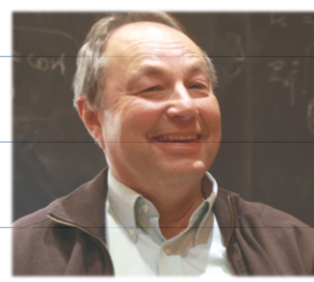
...unless we modify the initial theory,

$$S_{\Lambda_0}[\phi] \overset{(!!!)}{\rightsquigarrow} S_{\Lambda_0}[\phi] + S_{ct}[\phi; \Lambda_0].$$

We tune the g_i^R by hand to ensure that the renormalized couplings in S_Λ remain finite as $\Lambda_0 \longrightarrow \infty$.

What if the low-energy theory needs to contain g_i^I? Such theories cannot be valid as $\Lambda_0 \longrightarrow \infty$, which suppresses them. New physics is required!





4.1 Epilogue: Polchinski's Implementation of Wilsonian Renormalization

Q: How to actually compute in this framework?

A: By being very clever.

Recall that we defined $Z_\Lambda = \int_0^\Lambda D\phi_s \, e^{-S_\Lambda^{\text{eff}}[\phi_s]}$,

with

$$e^{-S_\Lambda^{\text{eff}}[\phi_s]} \equiv \int_\Lambda^{\Lambda_0} D\phi_f \, e^{-S_{\Lambda_0}[\phi_s+\phi_f]} \overset{B}{=}$$

$$\overset{B}{=} e^{-S^0_{\Lambda_0}[\phi_s]} \int_\Lambda^{\Lambda_0} D\phi_f \, \exp\left(-S^0_{\Lambda_0}[\phi_f] - S^{\text{int}}_{\Lambda_0}[\phi_s+\phi_f]\right).$$

To do this perturbatively, we need the propagator

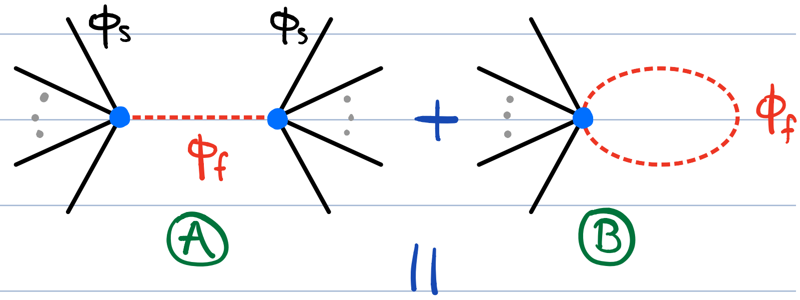
$$D_\Lambda(x-y) = \int_{\Lambda<|p|<\Lambda_0} \frac{d^d\vec{p}}{(2\pi)^d} \frac{e^{ip(x-y)}}{p^2+m^2} = \frac{1}{(2\pi)^d} \int_\Lambda^{\Lambda_0} dp \int_{S^{d-1}} d\Omega_{d-1} \, p^{d-1} \frac{e^{ip(x-y)}}{p^2+m^2}.$$

Flow infinitesimally: take $\Lambda_0 \rightsquigarrow \Lambda = \Lambda_0 - \delta\Lambda$, with $0 < \delta\Lambda \ll 1$. The propagator becomes

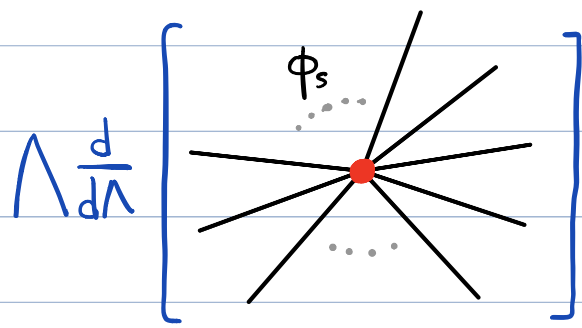
$$D_\Lambda(x-y) \simeq \frac{1}{(2\pi)^d} \frac{\Lambda^{d-1}\delta\Lambda}{\Lambda^2+m^2} \int_{S^{d-1}} d\Omega_{d-1} \, e^{i\Lambda\hat{p}(x-y)}.$$

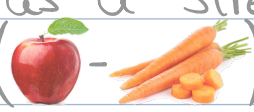
At order $\delta\Lambda$, the diagrams contributing to S^{eff}_Λ are:

- ϕ_s only or ϕ_f only: absorb the former into S^{eff}_Λ, the latter into vacuum energy.
- <u>one</u> ϕ_f propagator; treat ϕ_s as external:



Integrating out high energies amounts to zooming out until heavy propagators cannot be resolved, and D_Λ becomes an effective vertex.



[This may be viewed as a strengthened, differential form of the A-C ( -) decoupling theorem.]

Polchinski obtained an RG flow equation for S^{int}_Λ:

$$-\Lambda \frac{\partial}{\partial\Lambda} S^{\text{int}}_\Lambda[\phi] = \int_{\mathbb{R}^d\times\mathbb{R}^d} d^dx\, d^dy \Big[\underbrace{D_\Lambda(x-y) \frac{\delta S^{\text{int}}_\Lambda}{\delta\phi(x)} \frac{\delta S^{\text{int}}_\Lambda}{\delta\phi(y)}}_{\text{A}} - \underbrace{D_\Lambda(x-y) \frac{\delta^2 S^{\text{int}}_\Lambda}{\delta\phi(x)\,\delta\phi(y)}}_{\text{B}} \Big].$$

Now introduce the RG time $t \equiv \log \Lambda$.
Polchinski's equation can be rewritten as a heat flow:

$$\frac{\partial}{\partial t}\left(e^{-S_\Lambda^{int}[\varphi]}\right) = -\int_{\mathbb{R}^d \times \mathbb{R}^d} d^d x \, d^d y \, D_\Lambda(x-y) \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)} \left(e^{-S_\Lambda^{int}[\varphi]}\right).$$

Equivalently, $\left(\frac{\partial}{\partial t} + \Delta\right) e^{-S_\Lambda^{int}[\varphi]} = 0$, [Heat equation on $\mathbb{R} \times \{\text{fields}\}$]

where the Laplacian Δ on the space of fields is

$$\Delta \equiv \int_{\mathbb{R}^d \times \mathbb{R}^d} d^d x \, d^d y \, D_\Lambda(x-y) \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)} .$$

→ Last word: in AdS/CFT, the RG time t becomes the radial (bulk) direction z of AdS.

One says that holography "geometrizes" the RG.

The CFT_d living on $\partial(AdS_{d+1})$, i.e. at $z=0$, is a fixed point; the zz Einstein eq$^{\underline{n}}$ in first-order form takes the form of an RG flow equation. ö

Lore: UV in the bulk $\longleftrightarrow$ IR on the boundary;
IR in the bulk $\longleftrightarrow$ UV on the boundary.

Conclusions: meditate on the following.

1. the characterization of (ir)relevant operators;
2. the c-theorem and RG flows near fixed points;
3. the focusing of RG flows into universality classes;
4. the realization of RG flow as diffusion.

∴ RG flow reduces the number of degrees of freedom in a QFT. This decoupling or rearrangement of effective interactions is what makes physics possible.

Seeing renormalization for the first time

Introducing your seventh UV cutoff in the last two hours

When your theory isn't conformally invariant

A. A Scaling Argument

S_{Λ_0} and S_Λ^{eff} have different cutoffs: $\Lambda = s\Lambda_0 < \Lambda_0$, where $s \lesssim 1$. Compare them by rescaling:

$$x' \equiv sx, \qquad d^d x' = s^d d^d x, \qquad \partial'_\mu = \frac{1}{s}\partial_\mu,$$

$$p' = \frac{p}{s}, \qquad \Lambda' = \frac{1}{s}\Lambda = \Lambda_0, \qquad e^{ipx} = e^{ip'x'}.$$

But recall that the LSZ machine requires the kinetic term in S_Λ^{eff} to be scale-invariant:

$$S_\Lambda^{\text{eff}}[\phi] = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \sum_{i=1}^{n} g_i(\Lambda) \mathcal{O}_i(\phi, \partial\phi)\right] =$$

$$= \int d^d x' \left[\frac{1}{2} s^{2-d} (\partial'\phi(sx))^2 + \cdots\right] \overset{!}{=} [\text{LSZ}]$$

$$= \int d^d x \left[\frac{1}{2}(\partial'\bar{\phi})^2 + \cdots\right] \Longrightarrow \boxed{\bar{\phi}(x) \equiv s^{\frac{2-d}{2}} \phi(sx).}$$

The scaling behavior of the $\mathcal{O}_i$ can be inferred:

$$S_\Lambda^{\text{eff}} \supset \int d^d x \, g_i(\Lambda) \mathcal{O}_i(\phi, \partial\phi) =$$

$$= \int d^d x' \, g_i(\Lambda_0) s^{n_\phi\left(\frac{d-2}{2}\right) - n_\partial} \mathcal{O}_i(\bar{\phi}, \partial'\bar{\phi}).$$

We insist that this scaling be absorbed into the g_i:

$$\overline{g}_i(\Lambda) = s^{\Delta_{\mathcal{O}_i}} g_i\left(\frac{\Lambda}{s}\right), \qquad \boxed{\Delta_{\mathcal{O}_i} = n_\varphi\left(\frac{d-2}{2}\right) - n_\partial .}$$

$\boxed{\text{B.}}$ The Effective Action

Take $S_{\Lambda_0}[\phi] = \int_{\mathbb{R}^d} d^dx \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \mathcal{L}_{\text{int}}(\phi)\right]$

and decompose $\phi(x) = \phi_s(x) + \phi_f(x)$. Then, since the slow and fast modes are orthogonal, we find

$$S_{\Lambda_0}[\phi] = \int_{\mathbb{R}^d} d^dx \left[\frac{1}{2}(\partial\phi_s)^2 + \frac{1}{2}(\partial\phi_f)^2 + \frac{1}{2}m^2\phi_s^2 + \frac{1}{2}m^2\phi_f^2 + \mathcal{L}_{\text{int}}(\phi_s+\phi_f)\right] =$$

$$= S^0_{\Lambda_0}[\phi_s] + S^0_{\Lambda_0}[\phi_f] + S^{\text{int}}_{\Lambda_0}[\phi_s+\phi_f] \Longrightarrow$$

$$e^{-S_{\Lambda_0}[\phi_s+\phi_f]} = e^{-S^0_{\Lambda_0}[\phi_s]} \exp\left(-S^0_{\Lambda_0}[\phi_f] - S^{\text{int}}_{\Lambda_0}[\phi_s+\phi_f]\right).$$

Here Endeth the Talk.